Enumeration of polygon dissections with prescribed conditions

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Introduction

Motivation: Triangulations

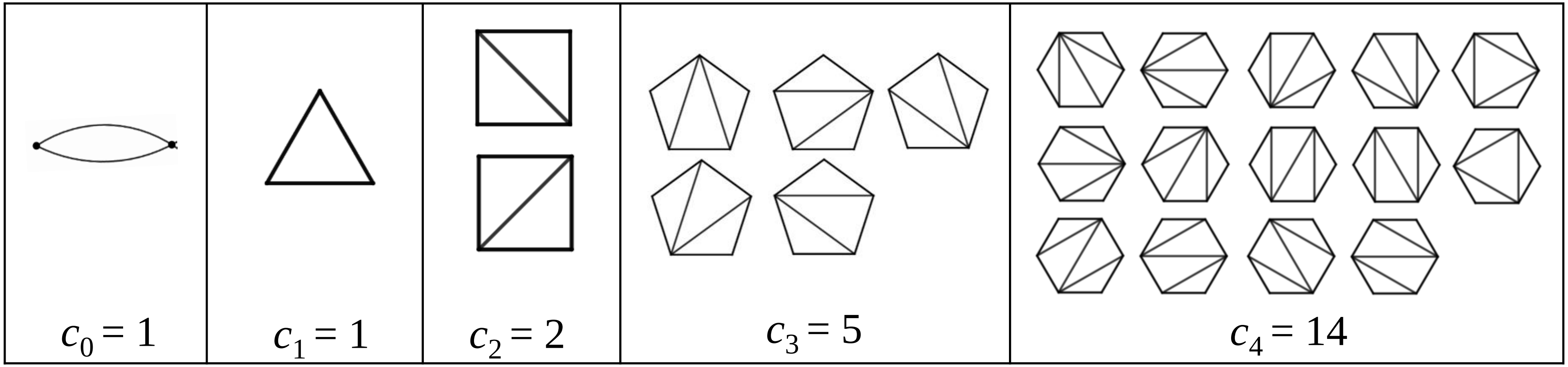


Fig.1: Triangulations

- Catalan Numbers: $c_n = \frac{1}{n+1}\binom{2n}{n}, \; n = 0, 1, 2, 3, \ldots$

Our problem and goal:

- How many ways to dissect a convex polygon into a given set of prescribed polygons?
- In this project, we completely solve the problem. (Main Theorem in Conclusion)

Notation

- Let $a_n(k_1^{p_1}, k_2^{p_2}, \ldots, k_t^{p_t}, 1^*)$ be the number of ways to dissect an $(n+2)$-gon into p_i copies of (k_i+2)-gons $(i = 1, 2, \ldots, t)$ and $n - \sum_{i=1}^{t} k_i p_i$ triangles, where $k_i \geq 2$ and $n - \sum_{i=1}^{t} k_i p_i \geq 0$.

- Examples:

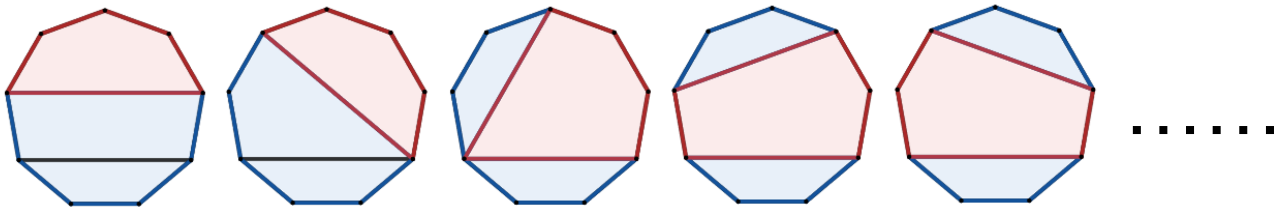


Fig. 2: $a_7(2^2, 3, 1^0) = 45 \to$ Dissect a 9-gon into 2 quadrilaterals, 1 pentagon, 0 triangle.

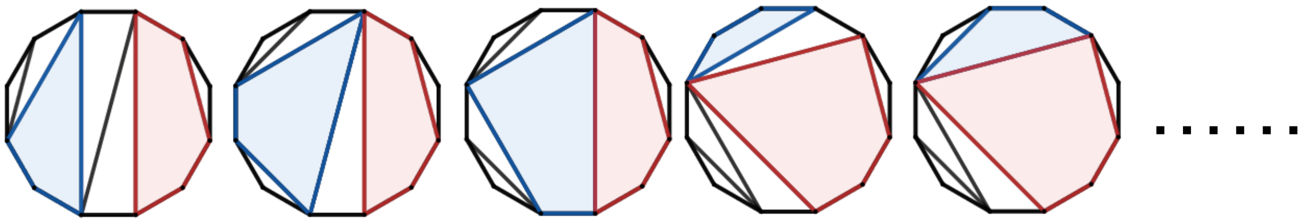


Fig. 3: $a_{10}(2, 3, 1^5) = 74256 \to$ Dissect a 12-gon into 1 quadrilateral, 1 pentagon, 5 triangles.

Generating function

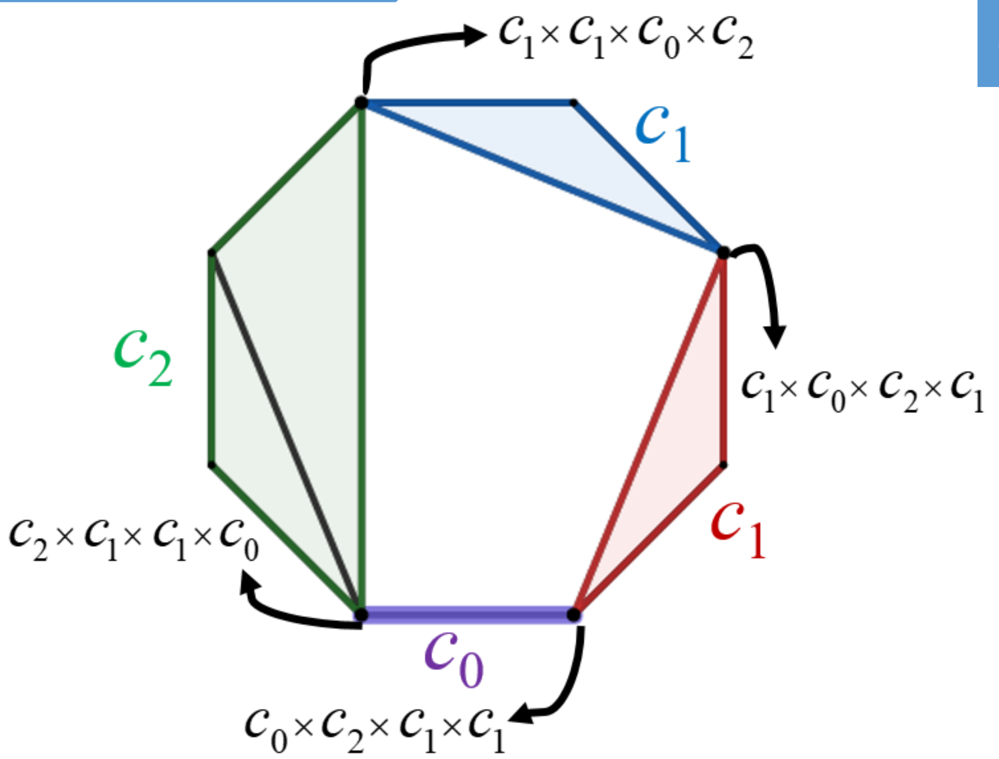
- In order to count $a_n(k_1,k_2,\ldots,k_t,1^*)$, we investigate its generating function $f_{(k_1,k_2,\ldots,k_t)}(x)=\sum_{n=0}^{\infty} a_n(k_1,k_2,\ldots,k_t,1^*)x^n$.
- For convenience, we denote $[x^n]g(x)$ is the coefficient of x^n of a generating function $g(x)$.

Generating function for Catalan numbers

- From our definition, $a_n(1^*) = c_n$.
- Let $C(x)=\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$ be the generating function for the Catalan Numbers.
- **Lemma 1:** $$\frac{C(x)^{k+1}-C(x)^k}{x}=C(x)^{k+2}$$
- **Lemma 2:** $$C(x)^k=\sum_{n=0}^{\infty}\frac{k}{n+k}\binom{2n+k-1}{n}x^n$$

Findings I : Dissections into distinct polygons

$a_n(k,1^*)$



$$a_n(2,1^{n-2}) = \frac{n+2}{4}[x^{n-2}]C(x)^4$$

Fig. 4: An example for a dissection of $k = 2$

Proposition 1 (Dissect into a single $(k+2)$-gon and triangles)

$$a_n(k,1^*) = \frac{n+2}{k+2}[x^{n-k}]C(x)^{k+2} = \binom{2n-k+1}{n+1}$$

- Every vertex on the $(k+2)$-gon can be associated with a string, each of which has $(k+2)$ Catalan numbers in it.
- In addition, the same string may start from any of the $(n+2)$ vertices.
- Multiplying the Catalan Numbers in each string and adding them together becomes the coefficient of x^{n-k} in $C(x)^{k+2}$.
- By previous lemmas, we have

$$f_k(x) = \sum_{n=0}^{\infty} a_n(k,1^*)x^n = \frac{1}{x}\cdot\frac{d}{dx}\left(\frac{x^{k+2}C(x)^{k+2}}{k+2}\right).$$

$a_n(k_1,k_2,1^*)$

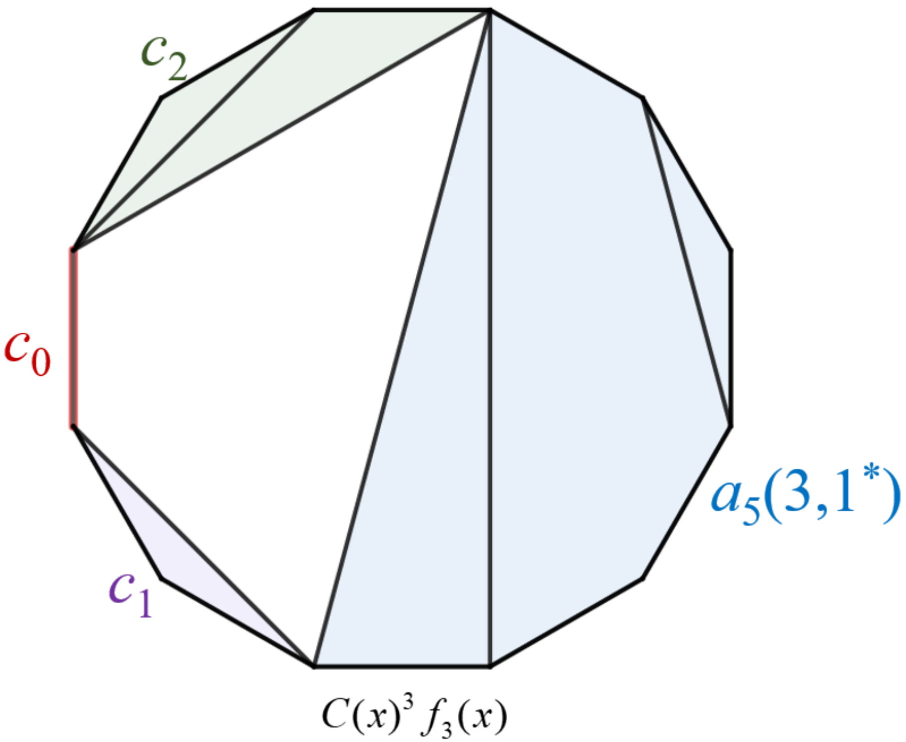


Fig. 5: An example for a dissection of $k_1 = 2$, $k_2 = 3$

Proposition 2 (Dissect into a single $(k_1 + 2)$-gon; a single $(k_2 + 2)$-gon; and triangles)

$$a_n(k_1,k_2,1^*) = (n+2)\binom{2n-k_1-k_2+2}{n+2}$$

- Let the $(k_1 + 2)$-gon be the main structure of the dissection.
- The other regions are $(k_1 + 1)$ triangulation regions and a region composed of a $(k_2 + 2)$-gon and triangles, which can be calculated by $f_k(x)$.
- We have $f_{(k_1,k_2)}(x) = \frac{1}{x}\cdot\frac{d}{dx}\left(x^{k_1+2} f_{k_2}(x) C(x)^{k_1+1}\right)$.

$a_n(k_1,k_2,k_3,1^*)$

Proposition 3 (Dissect into a single (k_1+2)-gon; a single (k_2+2)-gon; a single (k_3+2)-gon; and triangles)

$$a_n(k_1,k_2,k_3,1^*) = (n+2)(n+3)\binom{2n-k_1-k_2-k_3+3}{n+3}$$

(1)

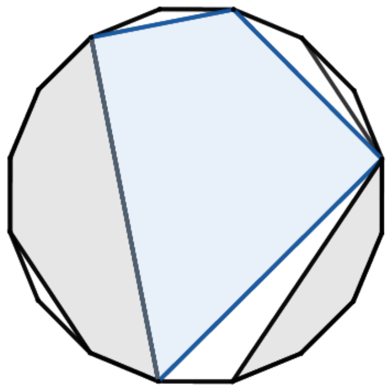


Fig. 6: Gray polygons in different regions

(2)

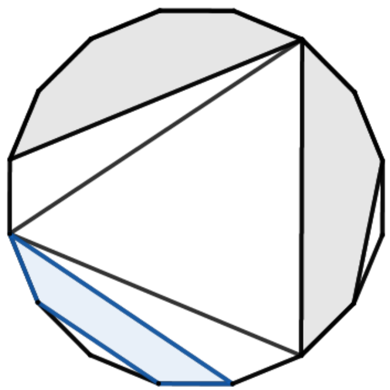


Fig. 7 : Gray polygons in common region

- The dissections should be classified according to the relative positions of the polygons when the $(n+2)$-gon is dissected into three or more polygons and a few triangles.
- $(n+2)\dfrac{(k_1+2)(k_1+1)}{k_1+2}[x^n]\left(x^{k_1} f_{k_2}(x) f_{k_3}(x) C(x)^{k_1}\right)$ (1)
- $(n+2)[x^n]\left(x^{k_1} f_{(k_2,k_3)}(x) C(x)^{k_1+1}\right)$ (2)
- From (1) and (2), we have

$$f_{(k_1,k_2,k_3)}(x) = \frac{1}{x}\cdot\frac{d}{dx}\left\{\begin{pmatrix}(k_1+1)x^{k_1} f_{k_2}(x) f_{k_3}(x) C(x)^{k_1} \\ +x^{k_1} f_{(k_2,k_3)}(x) C(x)^{k_1+1}\end{pmatrix} x^2\right\}$$

- From the result above we have our key lemma.

Idea: Shift

Key Lemma

The number of ways to dissect a given convex $(n+2)$-gon into a collection of $\{k_1+2, k_2+2, \ldots, k_t+2\}$-gons is the same as that to dissect into a collection of $\{h_1+2, h_2+2, \ldots, h_t+2\}$-gons, if $\sum_{i=1}^{t} k_i = \sum_{i=1}^{t} h_i$. That is, when $K = \sum_{i=1}^{t} k_i = \sum_{i=1}^{t} h_i$,

$$a_n(k_1, \ldots, k_t, 1^*) = a_n(h_1, \ldots, h_t, 1^*).$$

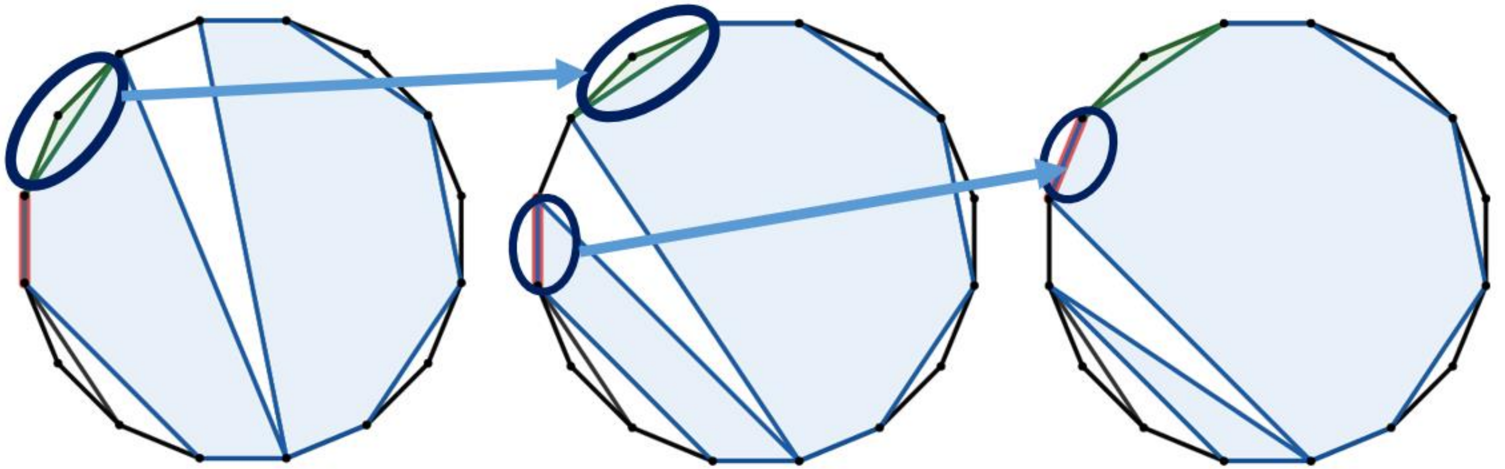


Fig.8: Shift of dissections

- Assign the unique region that is adjacent to the (k_i+2)-gon and move it cross the common region between the (k_i+2)-gon and the (k_j+2)-gon.
- The (k_i+2)-gon will be transformed into a (k_i+1)-gon, and the (k_j+2)-gon becomes a (k_j+3)-gon.

Specific triangles

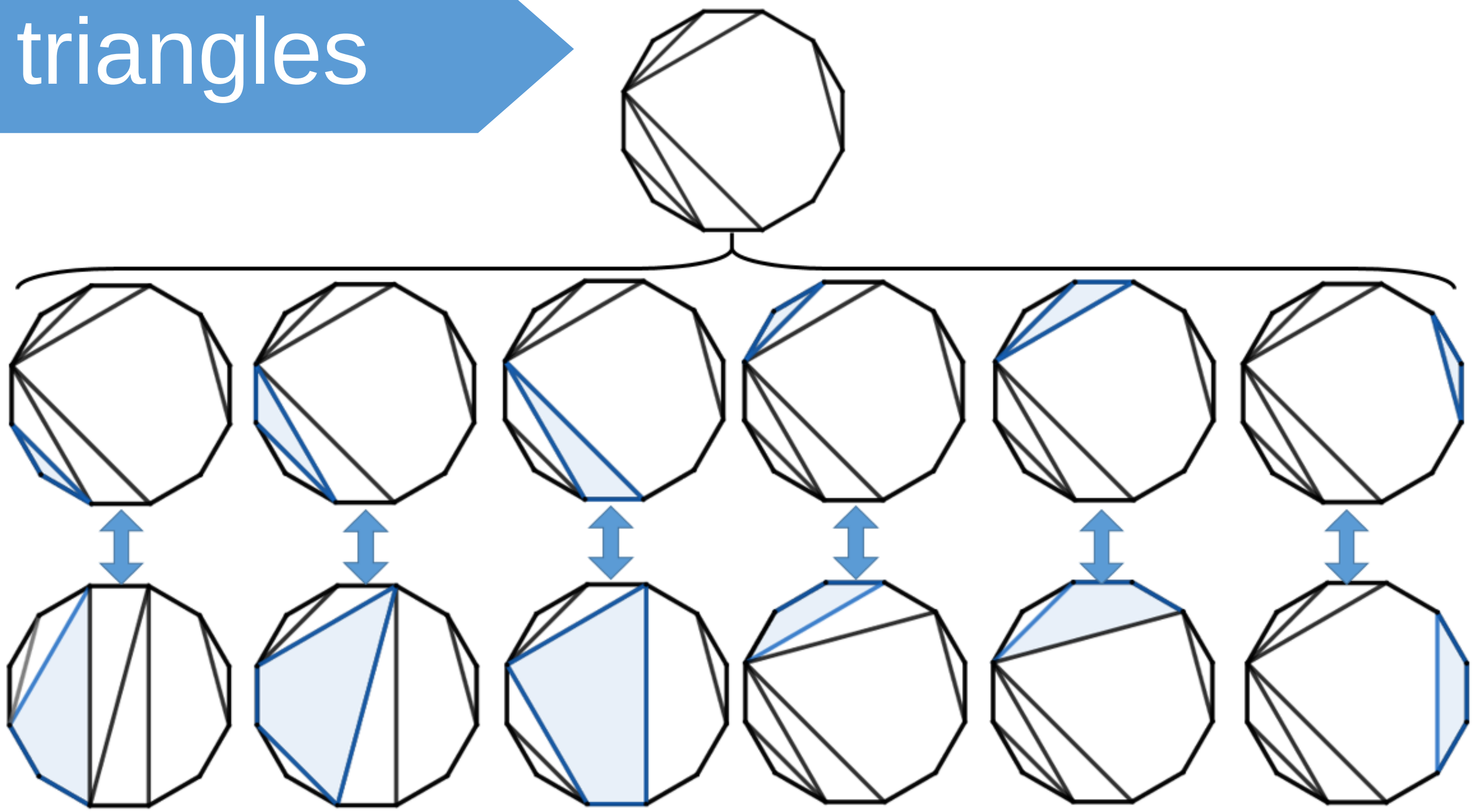


Fig. 9: An examples of the correspondence

- Transform polygons into “specific triangles” by shifting.
- The number of ways can be calculated by $a_n(k,1^*)$ obtained previously.

$$a_n(k_1,k_2,\ldots,k_t,1^{n-K}) = \frac{(n-K+t-1)!}{(n-K)!} \times a_n(K-t+1,1^{n-K+t-1})$$

$$= \frac{(n+t)!}{(n+1)!} \binom{2n-K+t}{n+t}$$

Key Proposition (Dissect into a collection of distinct polygons and triangles)

The number of way to dissect an $(n+2)$-gon into (k_i+2)-sides polygons $(i = 1, 2, \ldots, t)$ with distinct k_i's , that is $p_i = 1$ for all i, and $n - K$ triangles is $\frac{(n+t)!}{(n+1)!}\binom{2n-K+t}{n+t}$, where $\sum_{i=1}^{t} k_i p_i = K$.

- This formula can be seen as a generalization of Catalan Numbers:

Number of different types of polygons	**0**	**1**	**2**	…	***t***
Number of dissections	$\frac{1}{n+1}\binom{2n}{n}$	$\binom{2n-k+1}{n+1}$	$(n+2)\binom{2n-k_1-k_2+2}{n+2}$	…	$\frac{(n+t)!}{(n+1)!}\binom{2n-K+t}{n+t}$

Table.1: $a_n(k_1, k_2, \ldots, k_t)$

Findings II: Repeated polygons

- Suppose we want to dissect into a collection having p_i copies of $(k_i + 2)$-gon, $p_i \geq 2$.
- We can fix any one of the p_i copies and treat the remaining as distinct ones.
- The counting method previously analyzed for distinct polygons can then be applied, but the result would be $(p_i!)$ times more than the actual number.

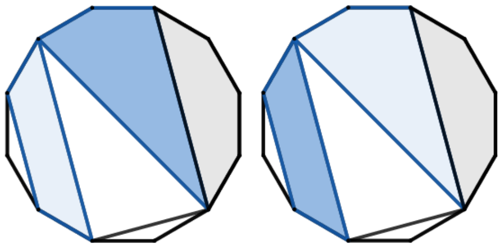


Fig. 10: An example for a dissection of $k_1 = 2,\ p_1 = 2,\ k_2 = 3$

$$a_n(k_1^{\ 2}, k_2, 1^*) = \frac{(n+2)(n+3)}{2!}\binom{2n-2k_1-k_2+3}{n+3}$$

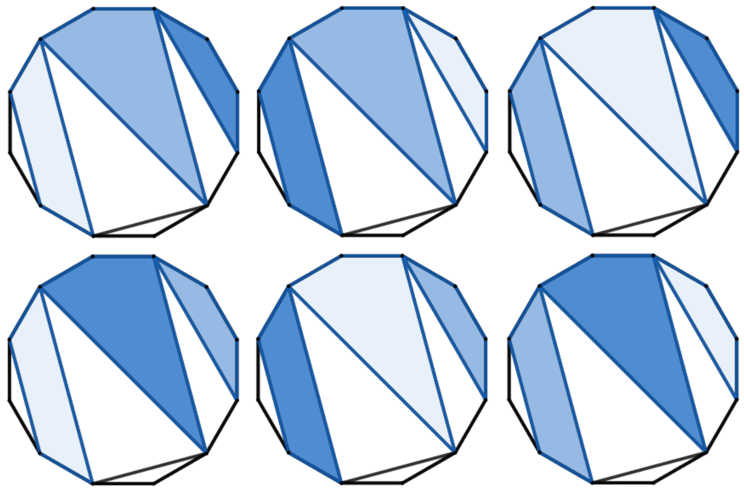


Fig.11: An example for a dissection of $k_1 = 2,\ p_1 = 3$

$$a_n(k^3, 1^*) = \frac{(n+2)(n+3)}{3!}\binom{2n-3k+3}{n+3}$$

Conclusion

- By the arguments above, we can solve the problem with **arbitrarily** prescribed conditions:

Main Theorem

The number of ways to dissect a convex $(n+2)$-gon into p_i copies of (k_i+2)-gons $(i=1,2,\ldots,t)$, and $n-K$ triangles, where $K=\sum_{i=1}^{t} k_i p_i$, is

$$a_n(k_1^{p_1}, k_2^{p_2}, \ldots, k_t^{p_t}, 1^{n-K}) = \frac{1}{\prod_{i=1}^{t}(p_i!)} \frac{(n+\sum_{i=1}^{t} p_i)!}{(n+1)!} \binom{2n-K+\sum_{i=1}^{t} p_i}{n+\sum_{i=1}^{t} p_i}.$$

References

[1] **Davis,T.** (2016). *Catalan Numbers*. Retrieved from http://www.geometer.org/mathcircles/, 2-12.

[2] **Stanley,R.P.** (2015). *Catalan numbers.* London: Cambridge University Press.

[3] **Tucker,A.** (2012). *Applied combinatorics.* NY: John Wiley & Sons Inc,308-315.

[4] **Wilf,H.** (1994) . *Generatingfunctionology*. Retrieved from https://www2.math.upenn.edu/~wilf/gfology2.pdf, 3-72.